

## Evaluation of Activation Parameters for a First Order Reaction from one Kinetic Experiment. Theory, Numerical Methods and Computer Program

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Computational methods are developed to determine activation parameters for a first order reaction from data obtained from one kinetic experiment with varying temperature. The methods have been programmed for a digital computer and tested on synthetic and experimental data.

Different equations for the description of the temperature dependence of the rate constants are investigated. A new equation which is a natural extension of the Arrhenius equation is proposed. This equation is shown to approximate experimental data as well or better than the commonly used equation.

Recently one of us<sup>1</sup> developed an experimental method which accurately estimates the activation parameters  $\Delta H^\ddagger$ ,  $\Delta S^\ddagger$ , and  $\Delta c_p^\ddagger$  for a reaction from only a single kinetic experiment. The method (called Varytemp) is based on the fact that the temperature can be varied during the experiment without loss of accuracy. By fitting a model which incorporates the temperature dependence of the reaction rate to the experimental data, one obtains estimates for the activation parameters. In this investigation, the numerical and computational details of the application of the Varytemp method, to a first order reaction are given.

In connection with the development of the numerical methods for Varytemp, it was of interest to investigate the theoretical models describing the temperature dependence of a rate constant. The Arrhenius equation can be extended in different ways to include the temperature dependence of the activation energy. Analogous equations can be also derived from the transition state theory.

Since the different equations are not theoretically or numerically equivalent, it is important to choose an equation which fits experimental data best and has good numerical properties. In this investigation, two such equations are compared. One, commonly used, is derived from the transition state theory

with certain assumptions on the temperature dependence of  $\Delta c_p^\ddagger$ . The other, proposed here, is purely empirical, but a natural extension of the Arrhenius equation.

## THEORY

The rate constant of a reaction ( $k$ ) is generally a function of the temperature ( $T$ ). Classically, this dependence has been described by the empirical Arrhenius equation.<sup>2</sup>

$$\ln k = A' + B'/T \quad (T \text{ }^\circ\text{K}) \quad (1)$$

This is better written as

$$\ln k = A + B(1/T - 1/T_0) + \varepsilon_k \quad (2)$$

to symbolize that the equation is subject to an error of measurement ( $\varepsilon_k$ ) and that the parameters  $A$  and  $B$  are valid only in an interval around the temperature  $T_0$  ( $^\circ\text{K}$ ).

The parameters  $A$  and  $B$  were given theoretical meaning by the transition state theory,<sup>3-5</sup> which defines

$$\ln k = \ln(kT/h) - \Delta H^\ddagger/RT + \Delta S^\ddagger/R \quad (3)$$

Here  $\Delta H^\ddagger$  is the enthalpy of activation,  $\Delta S^\ddagger$  the entropy of activation,  $k$  and  $h$  the constants of Boltzmann and Planck, respectively,  $R$  the gas constant and  $T$  the absolute temperature in degrees Kelvin.

Differentiation of eqn. 3 with respect to  $1/T$  gives (with use of Kirchhoff's

$$\text{law } \frac{\partial \Delta H^\ddagger}{\partial T} = T \frac{\partial \Delta S^\ddagger}{\partial T})$$

$$\frac{\partial \ln k}{\partial(1/T)} = -\frac{1}{R} (\Delta H^\ddagger + RT) \quad (4)$$

Thus we find for the parameters in eqn. 2

$$A = \ln(kT_0/h) - \Delta H_0^\ddagger/RT + \Delta S_0^\ddagger/R \quad (5)$$

$$B = -\frac{1}{R} (\Delta H_0^\ddagger + RT_0) = -\frac{1}{R} \cdot E_a^* \quad (6)$$

where the zero subscripts indicates temperature  $T_0$ .  $E_a^*$  is the Arrhenius activation energy.

With use of the eqns. 2, 5, and 6 we can obtain estimates for the derived parameters  $\Delta H^\ddagger$  and  $\Delta S^\ddagger$ . These are generally used for further discussions on the relations between reactivity and structure and so forth.

It must be emphasized, that the derived parameters,  $\Delta H^\ddagger$  and  $\Delta S^\ddagger$  estimated by fitting eqn. 2 to experimental data and then inserting the obtained parameters  $A$  and  $B$  into eqns. 5 and 6, are valid only as long as the transition state theory is valid.

Most kinetic data can adequately be described by eqn. 2. However, with more accurate measuring techniques, the accuracy of the rate constants becomes better, deviations from eqn. 2 start to show up, and the need for an

extension of the Arrhenius equation becomes evident. This extension can be made using two basically different principles.

From the current theory (in this case, the transition state theory), a probable and suitable extension can be derived, which is then used to discuss empirical data.

The other way is to start from experimental data and devise an extension which best fits the observations, without worrying about the theoretical meaning of this extension until later.

Robertson and Hyne<sup>6,7</sup> made the plausible assumption that the enthalpy of activation  $\Delta H^\ddagger$ , varies with temperature in a simple way.

$$\Delta H^\ddagger = a + bT + cT^2 \quad (7)$$

The insertion of this equation in the transition state theory gives

$$\ln k = A' + B'/T + C' \ln T + D'T \quad (8)$$

or better, referring to the temperature  $T_0$  and experimental situations.

$$\ln k = A + B(1/T - 1/T_0) + C \ln(T/T_0) + D(T - T_0) + \varepsilon_k \quad (9)$$

Here, the third parameter  $C$  contains the linear temperature variation of  $\Delta H^\ddagger$  ( $\Delta c_p^\ddagger$ ), and  $D$  the temperature variation of  $\Delta c_p^\ddagger$ .

The fourth parameter is generally not included, no kinetic data have hitherto been found, which cannot be described by the simpler equation (assuming that  $\Delta c_p^\ddagger$  is constant)

$$\ln k = A + B(1/T - 1/T_0) + C \ln(T/T_0) + \varepsilon_k \quad (10)$$

This indeed seems to support the assumption that  $\Delta c_p^\ddagger$  varies very slowly with temperature for reactions in solution.

Eqn. 10 or transformations thereof has been used by most investigators<sup>6-8</sup> to describe the temperature variation of  $k$  when eqn. 2 has been insufficient. It will be shown below, however, that the use of eqn. 10 leads to some disadvantages and that other extensions of eqn. 2 are equally or better suited to handle experimental data.

The only example of an empirical approach to the extension of eqn. 2 known to the authors is the Taylor expansion of Ives and Marsden.<sup>9</sup> They studied the equilibrium constants ( $K$ ) of weak acids in water, and used an orthogonal polynomial in  $(T - T_0)$  to describe the temperature variation of  $\ln K$ . In principle

$$\ln K = A + B(T - T_0) + C(T - T_0)^2 + D(T - T_0)^3 + \dots \varepsilon_k \quad (11)$$

The great advantage with the empirical approach is, that no assumptions about the theory are needed. The results of the use of an empirical equation are thus time invariant, *i.e.* independent of the present form of the theory. The fundamental parameters ( $A, B, C \dots$ ) can then, if so wished, be interpreted in view of the current theory. This gives derived parameters of the type  $\Delta H^\ddagger$ ,  $\Delta S^\ddagger$ ,  $\Delta c_p^\ddagger$  and so forth.

Thus the expansion of Ives and Marsden has some advantage; it is also of general scope. The statement of Kohnstam,<sup>8</sup> that the temperatures have to be uniformly spaced in order to make the orthogonal expansion possible,

is erroneous.<sup>10</sup> The only disadvantage with eqn. 11 is that the number of terms needed to describe the variation of  $\ln k$  (of kinetic data) is larger (usually four or five) than the corresponding number of eqn. 10 (three terms). This leads to unnecessarily large estimated errors of the derived parameters at temperatures other than  $T_0$ , since each of these ( $\Delta H^\ddagger$ ,  $\Delta S^\ddagger$ ,  $\Delta c_p^\ddagger$ ) contains a number of the fundamental parameters  $A$ ,  $B$ ,  $C$ ,  $D$   $\dots$  etc.

In this investigation, another empirical extension of eqn. 2 is proposed.

$$\ln k = A + B(1/T - 1/T_0) + C(1/T - 1/T_0)^2 + \dots + \varepsilon_k \quad (12)$$

This is the natural extension of the Arrhenius equation if we regard the latter as a Taylor expansion in  $x = 1/T - 1/T_0$  truncated to two terms. When these two terms are insufficient to describe the variation of  $\ln k$ , the third, quadratic, term is included, giving eqn. 12 with three terms etc. Eqn. 12 has some advantage in comparison with eqn. 11. First, in almost all cases investigated by us, only three terms have to be included in eqn. 12 to describe the experimental data *versus* four or five terms in eqn. 11. Apart from the consequences on the estimated accuracy of the derived parameters discussed above, this indicates that eqn. 12 has a closer connection to the "true" equation than has eqn. 11.

Secondly, with the present form of the theory, eqn. 12 has a closer resemblance to this than has eqn. 11. This leads to an easier interpretation of the resulting parameters  $A$ ,  $B$ ,  $\dots$  etc. This interpretation in view of the transition state theory is made as follows:

The two parameters  $A$  and  $B$  are of course the same as in the Arrhenius equation leading to eqns. 5 and 6. Differentiation of eqn. 4 leads to

$$\frac{\partial^2 \ln k}{\partial(1/T)^2} = - \frac{1}{R} \left( \frac{\partial \Delta H^\ddagger}{\partial(1/T)} - RT^2 \right) = - \frac{1}{R} \frac{\partial \Delta E_a^*}{\partial(1/T)} \quad (13)$$

which gives

$$\frac{\partial^3 \ln k}{\partial(1/T)^3} = - \frac{1}{R} \left( \frac{\partial^2 \Delta H^\ddagger}{\partial(1/T)^2} + 2 RT^3 \right) = - \frac{1}{R} \frac{\partial^2 \Delta E_a^*}{\partial(1/T)^2} \quad (14)$$

Remembering that  $-T^2(\partial/\partial T) = \partial/\partial(1/T)$ , the first four parameters in eqn. 12 have the following theoretical meaning in the transition state theory

$$A = \ln k_0 = \ln(\kappa/h) + \ln T_0 - \Delta H_0^\ddagger/RT_0 + \Delta S_0^\ddagger/R \quad (15)$$

$$B = -1/R(\Delta H_0^\ddagger + RT_0) \quad (16)$$

$$C = -1/R \left( \frac{\partial \Delta H_0^\ddagger}{\partial(1/T)} - RT_0^2 \right) = + T_0^2/R(\Delta c_{p_0}^\ddagger + R) \quad (17)$$

$$D = -1/R \left( \frac{\partial^2 \Delta H_0^\ddagger}{\partial(1/T)^2} + 2 RT_0^3 \right) \quad (18)$$

The zero subscripts symbolize that the parameters are defined at the temperature  $T_0$ .

Eqn. 12 truncated to three terms thus leads to a totally different temperature dependence of  $\Delta H^\ddagger$  than does eqn. 10. Since  $\Delta c_p^\ddagger$  in all cases hitherto investigated<sup>8</sup> has been found to be independent of the temperature (in ac-

cordance with eqn. 9) and since eqn. 12 (with three terms) leads to a  $1/T^2$  dependence of  $\Delta c_p^\ddagger$ , eqn. 12 seems to be empirically less well founded. As will be seen below, however, eqn. 12, truncated to three terms, gives as good or better fit to experimental data than does eqn. 10. This clearly shows the justification of use of eqn. 12 for description of the temperature dependence of  $\ln k$ .

Furthermore, in comparison with eqn. 10, eqn. 12 has great numerical advantages, especially in connection with the Varytemp method.

It has been pointed out,<sup>11,12</sup> that equations like eqn. 9 or 10 have bad approximation properties. This can easily be seen if the terms  $1/T-1/T_0$  and  $\ln(T/T_0)$  are expanded in power series.

$$1/T-1/T_0 = 1/(\Delta T + T_0) - 1/T_0 = \frac{1}{T_0} \left[ 1 / \left( \frac{\Delta T}{T_0} + 1 \right) - 1 \right]$$

Putting  $\Delta T/T_0 = z$  we get

$$1/T-1/T_0 = \frac{1}{T_0} (1/(z+1)-1) = -\frac{1}{T_0} (z - z^2 + z^3 - \dots) \quad (19)$$

$$\ln(T/T_0) = \ln \left( \frac{\Delta T + T_0}{T_0} \right) = \ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \dots \quad (20)$$

The third term in eqn. 10 is thus almost linearly dependent on the first and second terms and the dependence becomes worse as  $z$  decreases. If the terms were perfectly linearly dependent, the third term  $C \cdot \ln(T/T_0)$  could be expressed as a combination of the first two terms,  $A$  and  $B(1/T-1/T_0)$ . It is seen from the eqns. 19 and 20 that a combination of the three terms can be made so that the independent terms (the differences) are  $\frac{1}{2}z^2 - \frac{2}{3}z^3 + \dots$  etc. When  $z$  becomes small, these terms become unimportant and eqn. 10 becomes numerically unstable. The temperature variation of  $\ln k$  can then equally well be described by eqn. 2, and the third term in eqn. 10 becomes redundant.

If  $z$  is small enough, the results obtained from the application of eqn. 10 to experimental data becomes highly dependent on the distribution of the errors of measurement ( $\epsilon_k$ ), a totally unwanted situation.

In the experimental data,  $T$  is commonly larger than 300, while  $\Delta T$  (half the temperature interval) seldom is larger than 30. Thus  $z$  is smaller than 0.1 which necessitates an accuracy of  $k$  of about 2 or 3 % in order to make eqn. 10 numerically stable. In the Varytemp method, however, the temperature interval is commonly only 10 degrees and  $z$  is then of the order 0.01. The accuracy of the rate constants must then be about 0.1 %, which is hard to achieve.

It has thus been found impossible (also by actual trials) to use eqn. 10 in the Varytemp method. Instead we have incorporated eqn. 12 to describe the temperature dependence of  $\ln k$ .

## RESULTS

The performance of eqns. 10 and 12 (with three terms) was compared using synthetic and experimental data. The equations were fitted to the data using ordinary regression analysis (see numerical section) which also gave estimates for the standard deviation of  $\varepsilon$  ( $\sigma(\varepsilon)$ ) and the parameters  $\Delta H_0^\ddagger$ ,  $\Delta S_0^\ddagger$ , and  $\Delta c_p^\ddagger$ .

Synthetic data were generated in the following way. With  $\Delta H_0^\ddagger = 20.0$  kcal/mole  $\Delta S_0^\ddagger = -17.4$  e.u. and  $\Delta c_p^\ddagger = -35$  cal/deg mole (constant through the temperature interval),  $k$  values were computed from eqn. 3 for 14 different, equally spaced temperatures. To these theoretical  $k$  values a random, normally distributed (standard deviation  $\sigma \cdot k$ ) residual was then added to simulate the experimental situations. These data are similar to those obtained in the hydrolysis of methyl benzenesulphonate<sup>13</sup> (Table 2, run 1).

The synthetic data were then analysed according to eqns. 10 and 12. It is clearly seen (Table 1) that eqn. 10 gives a better fit to these data than does eqn. 12 (with three terms).

Table 1. Results of computations on synthetic data (generated according to eqn. 10) using eqn. 10 and eqn. 12 (three terms).  $T_0 = 298.16^\circ\text{K}$ .

| Number of runs | Generated $\sigma(\varepsilon)$ | Calculated $\sigma(\varepsilon)$ |                      | $\Delta H_0^\ddagger$ $\Delta H_0^\ddagger$ |         | $\Delta S_0^\ddagger$ $S_0^\ddagger$ |         | $\Delta c_p^\ddagger$ $\Delta c_p^\ddagger$ |         | Temperature interval $^\circ\text{K}$ |
|----------------|---------------------------------|----------------------------------|----------------------|---|---------|--------------------------------------|---------|---|---------|---------------------------------------|
|                |                                 | eqn. 10                          | eqn. 12              | eqn. 10                                     | eqn. 12 | eqn. 10                              | eqn. 12 | eqn. 10                                     | eqn. 12 |                                       |
| 1              | 0                               | $1.0 \times 10^{-8}$             | $1.3 \times 10^{-8}$ | 20.00                                       | 20.00   | -17.4                                | -17.4   | -35.0                                       | -38.5   | 65                                    |
| 3              | $10^{-5}$                       | $0.9 \times 10^{-8}$             | $1.3 \times 10^{-8}$ | 20.00                                       | 20.00   | -17.4                                | -17.4   | -35.0                                       | -38.5   | 65                                    |
| 3              | $10^{-4}$                       | $0.9 \times 10^{-8}$             | $1.3 \times 10^{-8}$ | 20.00                                       | 20.00   | -17.4                                | -17.4   | -35.0                                       | -38.5   | 65                                    |
| 3              | $10^{-3}$                       | $0.9 \times 10^{-8}$             | $1.4 \times 10^{-8}$ | 20.00                                       | 20.00   | -17.4                                | -17.4   | -34.9                                       | -38.4   | 65                                    |
| 3              | $3 \times 10^{-10}$             | $2.6 \times 10^{-8}$             | $2.8 \times 10^{-8}$ | 20.00                                       | 20.00   | -17.4                                | -17.4   | -35.4                                       | -38.9   | 65                                    |
| 3              | $10^{-3}$                       | $0.9 \times 10^{-8}$             | $0.9 \times 10^{-8}$ | 20.03                                       | 20.03   | -17.3                                | -17.3   | -36.0                                       | -39.7   | 65                                    |

The difference in goodness of fit is larger, the smaller is the standard deviation of  $\varepsilon$ . It is also seen that for this temperature interval and this  $\Delta c_p^\ddagger$  value, the relative accuracy in  $k$  has to be about 0.003 or better to make a discrimination between the models possible.

The differences in the  $\Delta c_p^\ddagger$  values obtained by the two equations are due to the fact that the reference temperature  $T_0$  is not the middle point of the temperature interval. With eqn. 10 we assume that  $\Delta c_p^\ddagger$  is constant in the interval whereas eqn. 12 gives a  $\Delta c_p^\ddagger$  which has a  $1/T^2$  temperature dependence (eqn. 18). This means that for temperatures lower than the middle temperature the  $\Delta c_p^\ddagger$  values computed by eqn. 12 will be algebraically larger than the  $\Delta c_p^\ddagger$  values computed by eqn. 10 and *vice versa*. However, the ordering of the  $\Delta c_p^\ddagger$  values of eqn. 12 does not change with temperature and hence the values of a reaction series can be compared at any desired temperature, as long as all parameters are computed at this temperature.

Table 2. Results of computations on experimental

| Run | Hydrolysis in water of<br>(Compound)    | $T_0$<br>°K | Temp.<br>interval<br>°K | Ref. | Eqn. 10<br>$\sigma \times 10^3$ | Eqn. 12<br>$\sigma \times 10^3$ |
|-----|---|-------------|-------------------------|------|---------------------------------|---------------------------------|
| 1   | Methyl benzene sulphonate               | 298.16      | 60                      | 13   | 2.46                            | 1.62                            |
| 2   | Ethyl benzene sulphonate                | 298.16      | 65                      | 13   | 2.38                            | 2.46                            |
| 3   | Isopropylbenzene sulphonate             | 298.16      | 35                      | 13   | 3.06                            | 3.10                            |
| 4   | Propyl benzene sulphonate               | 298.16      | 60                      | 13   | 2.69                            | 2.57                            |
| 5   | Equilibrium const. weak acid            | 298.16      | 40                      | 9    | .467                            | 0.357                           |
| 6   | <i>t</i> -Butylchloride                 | 298.16      | 20                      | 14   | 1.86                            | 1.88                            |
| 7   | Benzyl chloride                         | 323.16      | 50                      | 15   | 1.70                            | 1.29                            |
| 8   | Allyl chloride                          | 323.16      | 50                      | 15   | 2.30                            | 2.15                            |
| 9   | Allyl bromide                           | 323.16      | 50                      | 15   | 2.29                            | 2.80                            |
| 10  | Allyl iodide                            | 323.16      | 45                      | 15   | 3.69                            | 3.67                            |
| 11  | 2,2-Dichloropropane                     | 298.16      | 30                      | 16   | 4.30                            | 4.29                            |
| 12  | 2,2-Dibromopropane                      | 298.16      | 25                      | 16   | 3.73                            | 3.67                            |
| 13  | 2-Bromo-2-chloropropane                 | 298.16      | 25                      | 16   | 1.08                            | 1.07                            |
| 14  | <i>t</i> -Pentyl chloride               | 278.16      | 12                      | 17   | 1.81                            | 1.83                            |
| 15  | 2-Chloro-2-methyl-propyl methyl ether   | 298.16      | 23                      | 17   | 18.1                            | 18.5                            |
| 16  | <i>t</i> -Butyl-dimethyl-sulphonium ion | 343.16      | 35                      | 17   | 1.0                             | 1.0                             |

Since the difference between the models becomes larger with an increasing  $\Delta c_p^\ddagger$  and an increasing temperature interval, it is not directly possible to quantitatively compare the results for other combinations of these factors. However, qualitative comparison can be made if one remembers this behaviour of the two equations.

Experimental data taken from the literature were then examined using the eqns. 10 and 12. The results are given in Table 2. It is seen that the derived values  $\Delta H_0^\ddagger$  and  $\Delta S_0^\ddagger$  differ very little (as expected from the results in Table 1) but that the  $\Delta c_p^\ddagger$  values differ substantially. Part of this variation is due to the choice of  $T_0$  for the different reactions, as discussed above, and part of it is due to the difference in models. It is seen that eqn. 12 generally estimates a smaller variation of  $\Delta c_p^\ddagger$  within a reaction series (*e.g.* runs 1–4, 6–10, 11–13) than does eqn. 10.

The most important result, however, is that eqn. 12 generally fits the data as well or better than eqn. 10. The only exception is run 9; however, the difference in this run is not large. This clearly shows that the use of eqn. 12 is justified.

#### COMPUTATIONAL ASPECTS

##### Determination of the rate constant at different temperatures

In all earlier investigations, the rate constants at different temperatures were determined separately from different kinetic runs. However, this method introduces errors due to factors which vary between the experiments such as weighing errors, volume errors and so forth. The precision within the different

data using eqn. 10 and eqn. 12 (three terms).

| Eqn. 10<br>$\Delta H_0^\ddagger$<br>kcal/mole | Eqn. 12<br>$\Delta H_0^\ddagger$<br>kcal/mole | Eqn. 10<br>$\Delta S_0^\ddagger$<br>e.u. | Eqn. 12<br>$\Delta S_0^\ddagger$<br>e.u. | Eqn. 10<br>$\Delta c_{p_0}^\ddagger$<br>cal/deg.mole | Eqn. 12<br>$\Delta c_{p_0}^\ddagger$<br>cal/deg.mole | Comment           |
|---|---|--|--|--|--|-------------------|
| 21.43 ± .03                                   | 21.43 ± .02                                   | -17.36 ± .10                             | -17.37 ± .06                             | -33.8 ± 1.9  | -36.7 ± 1.6  | eqn. 12<br>better |
| 22.01 ± .03                                   | 22.01 ± .03                                   | -15.69 ± .10                             | -15.69 ± .10                             | -34.0 ± 1.6  | -37.3 ± 1.7  | equal             |
| 21.83 ± .05                                   | 21.83 ± .05                                   | - 8.03 ± .17                             | - 8.03 ± .18                             | -40.6 ± 5  | -38.6 ± 5  | equal             |
| 21.48 ± .07                                   | 21.51 ± .06                                   | -18.00 ± .22                             | -17.90 ± .20                             | -30.9 ± 2.5  | -35.5 ± 2.6  | equal             |
| —   | —   | —  | —  | -29.5 ± .8   | -29.4 ± .6   | better            |
| 22.55 ± .13                                   | 22.60 ± .13                                   | 2.0 ± .4                                 | 2.2 ± .4                                 | -84 ± 9  | -76 ± 8  | equal             |
| 20.37 ± .03                                   | 20.37 ± .02                                   | -20.5 ± .1                               | -20.5 ± .1                               | -42.8 ± 2  | -40 ± 1.2  | better            |
| 22.00 ± .03                                   | 22.00 ± .03                                   | -19.8 ± .1                               | -19.8 ± .1                               | -49.8 ± 3  | -52.6 ± 3  | equal             |
| 21.19 ± .03                                   | 21.19 ± .04                                   | -17.2 ± .10                              | -17.2 ± .12                              | -59.6 ± 2  | -55.6 ± 2  | worse             |
| 22.64 ± .35                                   | 22.63 ± .35                                   | -15.2 ± 1                                | -15.2 ± 1                                | -43 ± 30   | -42 ± 30   | equal             |
| 27.00 ± .23                                   | 27.06 ± .23                                   | .8 ± .8                                  | 1.0 ± .8                                 | -75 ± 14   | -83 ± 15   | equal             |
| 26.93 ± .13                                   | 26.94 ± .12                                   | 3.8 ± .4                                 | 3.8 ± .4                                 | -80 ± 16   | -84 ± 16   | equal             |
| 25.94 ± .02                                   | 25.94 ± .02                                   | 3.2 ± .1                                 | 3.2 ± .1                                 | -94 ± 2  | -92 ± 4  | equal             |
| 22.49 ± .06                                   | 22.49 ± .06                                   | 3.9 ± .2                                 | 3.9 ± .2                                 | -112 ± 32  | -113 ± 32  | equal             |
| 25.2 ± .2                                     | 25.2 ± .2                                     | - 4 ± .7                                 | - 5 ± .7                                 | -102 ± 40  | -100 ± 40  | equal             |
| 31.61 ± .01                                   | 31.61 ± .01                                   | 7.15 ± .03                               | 7.15 ± .03                               | - 7.5 ± 1.6  | - 7.3 ± 1.6  | equal             |

experiments is in general much better than the precision of the external procedures conducted before the experiments. Thus the accuracy of the activation parameters is determined mainly by the external errors, an unhappy situation since these errors are very hard to decrease.

The method Varytemp determines all the rate constants from the same kinetic run, using varying temperature. Since the rate constants estimated by this method all have the same external errors and since constant errors are unimportant in the computation of the activation parameters, the errors of the latter are much smaller than if the rate constants are determined separately.

In order to clarify the computational method, we will first give a survey of the theory behind the method of determination of rate constants from one kinetic experiment.

The theory will be shown applied to a first order reaction; however, it can easily be generalized to more complex reactions.

A first order reaction at constant temperature is described by the differential equation

$$dA/dt = -kA \quad (21)$$

where  $A$  is the concentration of the reactant,  $t$  is the time and  $k$  the rate constant. The reaction is recorded by measuring the variable  $y$  which is here assumed to be a linear function of  $A$

$$y = aA + c$$

The variable  $y$  is temperature dependent (without reaction),

$$y = y_0(1 - \gamma_2(T - T_0)) + c \quad (22)$$



This dependence is caused by volume expansion and other factors influenced by temperature. Thus

$$y = a_0 A_0 (1 - \gamma_2 (T - T_0)) + c \quad (23)$$

Differentiation of eqn. 23 with respect to time gives

$$\frac{dy}{dt} = - a_0 A_0 \gamma_2 \frac{dT}{dt} + \frac{dA_0}{dt} a_0 (1 - \gamma_2 (T - T_0)) \quad (24)$$

division by  $y - c$  (evaluated from eqn. 23) gives

$$\frac{dy}{dt} / (y - c) = \frac{dA_0}{dt} / A_0 - \gamma_2 \frac{dT}{dt} / [1 - \gamma_2 (T - T_0)] \quad (25)$$

and since

$$\frac{dA_0}{dt} / A_0 = -k$$

we finally have

$$-\frac{dy}{dt} / (y - c) - \frac{\gamma_2 (dT/dt)}{1 - \gamma_2 (T - T_0)} = k(T) \quad (26)$$

For pseudo first order reactions, the rate constant must be corrected for the volume dependent catalyst concentration. The left side of eqn. 26 is then multiplied by  $[1 + \gamma_1 (T - T_0)]$ , where  $\gamma_1$  is the volume expansion coefficient of the experimental solution.

$k(T)$  can thus be estimated from  $dy/dt$ ,  $y$ ,  $c$ ,  $\gamma_1$ ,  $\gamma_2$ ,  $T$ , and  $dT/dt$ .

To compute these quantities for different temperatures it is sufficient to observe  $y$  as a function of  $t$  with parallel observation of  $T$  as a function of  $t$ . From the observed data one can then compute  $dy/dt$  and  $dT/dt$  for each  $t$  value.

The quantities  $\gamma_1$  and  $\gamma_2$  have to be measured separately.

### Experimental data

The quantities necessary to compute the activation parameters by the Varytemp method are

1. A number of  $y$ -values with corresponding  $T$  and  $t$  values. ( $y_i$ ,  $T_i$ ,  $t_i$ ) symbolizes one such data triplet.
2. Numerical values for  $\gamma_1$  (the volume expansion coefficient of the experimental solution) and  $\gamma_2$ , the temperature dependence of  $y$ . The former parameter is easily measured with the required accuracy using an ordinary pycnometer; the latter parameter is most easily estimated using the Varytemp method excluding the catalyst. If this technique is impossible (*e.g.* the solvent catalyses the reaction), one can put  $\gamma_2 = \gamma_1$  without any error in the estimates. The correction involving  $\gamma_2$  is generally small (of the order of 0.001 to the values of  $k$ ) and almost constant in the interval and thereby affects only the entropy

term. Omitting  $\gamma_2$ , one obtains an error in the entropy of about 0.02 e.u. if the rate constant is of the usual order 0.01/min and about 0.25 e.u. if the rate constant is of the order 0.001/min in the experimental temperature interval.

### Smoothing of the experimental data and computation of derivatives

The theory of regression analysis demands the errors in the independent variables to be much smaller than the error in the dependent variable.<sup>18</sup> The precision of the time measurements is usually very good (better than 1/10 000) but the precision of the thermometers is generally not better than 0.005°C. If the temperature interval is about 10°C as in the investigations of Ahlberg,<sup>1</sup> the error in temperature cannot be neglected in comparison with the error in  $y$  (about 1/1000). To overcome this difficulty (which is purely theoretical) the  $T$ -values are smoothed, which decreases their error about 5 times.

The computations of derivatives and smoothed values ( $\bar{y}$  and  $\bar{T}$ ) are made by fitting third degree polynomials of  $t$  (time) to the observed  $y$  and  $T$  values. The estimated values  $\bar{y}=f(t)$ ,  $dy/dt=f'(t)$ ,  $\bar{T}=g(t)$  and  $dT/dt=g'(t)$  are subsequently used in the computations. Ordinary regression analysis<sup>18</sup> is used in the fitting, the computation of smoothed  $\bar{y}$  values and  $dy/dt$  is shown below.

For the  $k$ :th point these values are computed as follows. The observed  $y$  values  $y_i$  are assumed to be given by the equation

$$y_i = a_1 + a_2 z_i + a_3 z_i^2 + a_4 z_i^3 + e_i \quad \text{where } z_i = t_i - t_k \quad (27)$$

Only the points where  $k-15 \leq i \leq k+15$  are considered in the analysis, and thus 31 points are involved each time. The index of the 16:th point is  $k$ . The parameters  $a_1 - a_4$  are computed so that  $\sum_i w_i e_i^2$  becomes minimal in the usual way.<sup>18</sup>

The weighting function  $w_i$  is incorporated to make the computed residuals  $e_i$  have the distribution corresponding to the method of measurement of  $y$ . (In the cases where the present method has hitherto been used, the instrument has been a polarimeter where the errors are proportional to the measured values  $y$ , as long as  $|y| > 1$ , otherwise constant.  $y$  is the optical rotations in degrees.  $w_i$  thus has been given the value  $1/y_i^2$  where  $|y_i| > 1$ , otherwise 1).

The  $k$ :th point then gets the new  $y$  value  $a_1$  and the derivative  $dy/dt$  equals  $a_2$  since  $z_k = 0$ .

The values  $\bar{T}$  and  $dT/dt$  are analogously computed; however, the weighting functions used in these computations are equal to 1 since the thermometer accuracy is the same over the temperature interval.

In this way we only obtain estimates for  $\bar{y}$ ,  $dy/dt$ ,  $\bar{T}$  and  $dT/dt$  for  $k=16, 17, 18, \dots, n-15$ ; the first and last 15 points are therefore excluded from the further computations.

### Computation of the activation parameters

From the values of  $\bar{y}$ ,  $dy/dt$ ,  $\bar{T}$ , and  $dT/dt$ , the rate constant  $k(T)$  at each temperature is then computed from eqn. 26 and thus  $(k_i, \bar{T}_i)$  pairs are obtained for  $i=16, 17 \dots n-15$ . The final computations which give estimates for the activation parameters are then performed as follows.

1. *First approximation.* Initial values for further computations are obtained by regression analysis on eqn. 12. Better estimates, however, can be obtained by use of the nonlinear equation

$$k_i = \exp(A + Bx + Cx^2 + \dots) + e_i \quad (28)$$

where

$$x = 1/T - 1/T_0$$

since fitting of this equation to the  $k$  values gives the right error distribution and the variation of  $k$  is always larger than that of  $\log k$ . The difference in parameters between the analysis according to eqns. 12 and 28 is, however, small, never larger than 1 % in  $\Delta H^\ddagger$ , 2 % in  $\Delta S^\ddagger$ .

2. *Final computation.* Given good approximate values for  $A$ ,  $B$ , and  $C$ , the fitting of eqn. 28 to the  $k$  values is easily made with the non linear least squares method of Gauss.<sup>18</sup> In this method a truncated Taylor expansion of the theoretical function is used to get corrections to the approximate parameter values.

$$F(\alpha, x) \simeq F(\alpha_0, x) + \sum_k \frac{\partial F(\alpha_0, x)}{\partial \alpha_k} \partial \alpha_k \quad (29)$$

where  $\alpha$  symbolizes the parameter vector with the elements  $\alpha_k$ . In this problem we have

$$k(A, B, \dots) \simeq k(A_0, B_0, \dots) + \frac{\partial k}{\partial A} A + \frac{\partial k}{\partial B} \partial B + \dots \quad (30)$$

Differentiation of eqn. 28 gives

$$k(A, B, C, \dots) = k(A_0, B_0, C_0, \dots) \cdot [1 + \partial A + \partial B \cdot x + \partial C \cdot x^2 + \dots] \quad (31)$$

Fitting of this equation to the experimental  $k$  values thus gives corrections to  $A_0$ ,  $B_0$ , and  $C_0$ . These new estimates are then used as initial values for a new analysis to give new corrections *etc.* This iterative process is stopped when all corrections are smaller than a predetermined value, in this case  $10^{-6}$  times the corresponding parameter estimate.

The process is rapidly convergent, and in all cases tested the number of iterations has been less than 5.

In the above mentioned regression analysis on eqn. 28 a scaling of the  $x$  variable is necessary. Since  $1/T$  is of the order  $1/300$ ,  $1/T - 1/T_0$  is less than  $10^{-4}$  when the temperature interval is  $10^\circ\text{C}$ . The moment matrix to be inverted in the regression analysis contains sums of the type  $\sum x$ ,  $\sum x^2$ ,  $\sum x^3$ , and  $\sum x^4$ . These sums will then be of highly different order of magnitude, and since all computers have limited accuracy, spurious results will be generated if no

precautions are made. The problem is easily solved by a scaling of the variable  $x$ . Instead of  $x$  we use  $\xi = Sx$  in the computations. The scaling factor  $S$  is chosen to make  $\xi$  vary between plus and minus two in the interval, and thus the sums in the moment matrix now have the same order of magnitude. After the analysis, the resulting parameters are multiplied by  $S^r$  where  $r$  is the corresponding power of  $x$  in eqn. 28. The same technique is used in the approximate analysis of eqn. 12.

In the analysis of eqn. 28 no weighting function is used. This approximation is made since the error in  $dy/dt$  is proportional to  $|y|$ . When  $|y|$  is larger than 1, the error of  $dy/dt/(\bar{y}-c)$  is then approximately constant since the error in  $\bar{y}$  is small and the constant  $c$  is small.

### Error analysis

Since regression analysis is used throughout the computations, the accuracy of the computed parameters can easily be estimated in the traditional way.<sup>18</sup> If the estimated variance of  $k$  is  $\sigma^2$ , the estimated variance of the parameters  $A, B, C \dots$  is  $\sigma_j^2 = \sigma^2 H_{jj}$ , where  $H_{jj}$  are the diagonal elements of the covariance matrix obtained by inversion of the moment matrix. The estimated accuracy (on 95 % level) is then approximately  $2 \sigma_j$ .

With an accuracy of  $y$  of about 1 promille, and of  $T$  of about 0.01°C, 150 points in a temperature interval of 10°C give enthalpies of activation accurate to about 20–50 cal/mole and entropies of activation accurate to about 0.1–0.3 e.u. In order to get accurate heat capacities of activation, the temperature interval has to be increased to 20°C (with the present experimental accuracy) since the correlation between the  $Cx^2$  term and the  $A$  and  $Bx$  terms is too large in an interval of 10°C to give  $C$  values significantly different from zero. It is hoped, however, that refinement of the numerical methods will make it possible to obtain estimates for  $\Delta c_p^\ddagger$  using only an interval of 10°C.

### Computer program

The described numerical procedure has been programmed in 3200 Fortran and thoroughly tested on synthetic and experimental data.

The program accepts input either of the conventional form  $(k, T)$  pairs, or data obtained by the Varytemp method.

The computations are performed as outlined in the numerical section. Three different models are tested in each run. The  $k$  values are treated according to the model  $k = \exp(A + Bx + Cx^2 + Dx^3)$  where

1.  $C$  and  $D = 0$ , which corresponds to a heat capacity of activation of  $-R$  in the whole interval.
2.  $D = 0$ , which gives  $\partial \Delta H^\ddagger / \partial (1/T)$  a constant value in the interval.
3. All parameters are varied.

Afterwards the model that gives parameters which all are significantly different from zero should be chosen. A parameter is significantly different from

zero if the estimated value is larger than two times its estimated standard deviation ( $\sigma$ ).

A typical Varytemp run with 150 data triplets takes about 30 sec on a CDC 3200 computer.

In order to adapt the program to other experimental problems, a few changes have to be made. The weighting functions in the different regression analyses have to be modified in order to suit the actual error distribution. Also some changes in the input and output may be necessary to suit the form of data obtained in the experiments. Otherwise, the program should be suited to handle any problem giving kinetical data following eqn. 21.\*

#### Comparison with the conventional method

The Varytemp method has been tested on the pseudo first order rearrangement of 1-ethyl indene.<sup>1,19</sup> The Varytemp method gives the same results (within the estimated accuracy) as the conventional method but with much less work.

The rate constants estimated by the conventional method had an estimated accuracy of about 0.01 %, but a between experiment variability of about 0.5 %, in spite of the extreme caution taken to avoid external errors.<sup>1</sup> In order to get an accuracy of 50 cal/mole in the enthalpy of activation and 0.3 e.u. in the entropy of activation, one has to determine about 50 rate constants in a 10°C interval (with the observed between experiment variability).

The method of Varytemp gets parameters with this accuracy in one experiment (one test experiment has also to be made to assure that the reaction is strictly first order), and can thus be said to be very time saving or/and accurate.

#### DISCUSSION

During the last few years, a new situation has developed for the experimentalist. As large and fast computers have become accessible to almost everybody, the design of experiments can be radically changed.

Classically, the experiments have been conducted to give an output which is easy to interpret without lengthy computations and in organic chemistry most numerical work has consisted of making graphs on paper.

Today however, when calculations of enormous complexity are made by computers in seconds, computational difficulties are no longer any restraint on experiments. The experiments can now be designed in order to fulfil other criteria, *e.g.* save labour time and/or decrease experimental errors.

The development of the Varytemp method is a typical example. The aim was to obtain activation parameters with high accuracy. Since the errors in the results using the conventional method arise mainly from between experiment variability, which is hard to control, the natural way to obtain the wanted accuracy is to conduct only one experiment. This must then be designed to allow determination of the parameters directly; thus, the temperature must be varied during the experiment.

\* Program list and manual are available on request to S.W.

It has been shown in this investigation, that the Varytemp method indeed gives parameters with the desired accuracy.

Different models have been tested for the determination of activation parameters. The model adopted in this investigation is a natural extension of the Arrhenius equation and has good numerical properties. This model also gives as good or better fit to experimental data than does the model which is derived from the transition state theory, assuming a constant heat capacity of activation. The latter model is numerically unstable in small temperature intervals and therefore is impossible to use in the Varytemp method.

Since the two models give different estimates for the heat capacities of activation, a choice should be made as to which model should be used in future investigations. The results of this investigation indicate that use of eqn. 12 should be preferred, especially since eqn. 12 has the advantage of not being chained to the transition state theory. However, further studies on this subject must be made before definite choice is made. It is hoped that an expanded examination of the performance of the eqns. 10 and 12 in connection with experimental data will provide more insight into the problem.

The question as to which functions and parameters should be used to describe the temperature dependence of the rate of a chemical reaction is in our view not yet solved. It has been shown by Exner<sup>20</sup> that the parameters  $\Delta H^\ddagger$  and  $\Delta S^\ddagger$  are highly dependent on each other (this can be seen from eqn. 15). These parameters are thus less suited to describe variation between different reactions if this variation is small.

The parameters  $A$ ,  $B$ ,  $C$ , and  $D$  obtained by analysis of data according to eqn. 12 have the desired numerical properties. When a comparison with theoretical quantities is desired, however, we must still go *via* the conventional  $\Delta H^\ddagger$  and  $\Delta S^\ddagger$  parameters, which have the undesirable covariation.

Other parameters having the same theoretical foundation as  $\Delta H^\ddagger$  and  $\Delta S^\ddagger$  but better numerical properties are still to be invented.

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